Does Elko Spinor Field Imply the Existence of an Axis of Locality?

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Abstract

In this paper we show that the statement in reference Ahluwalia, Lee, and Schritt (2011) that the existence of elko spinor fields implies in an *axis of locality* is equivocated. The anticommutator $\{\Lambda(\mathbf{x},t),\Pi(\mathbf{x},t\}$ is strictly local.

1 The Integral Appearing in the Propagator

In [1] authors calculated the propagator for an elko spinor field supposed to satisfy the Klein-Gordon equation in Minkowski spacetime. They obtained a sum of two terms, the first being the usual propagator (fundamental solution) for a Klein-Gordon field and the second involves evaluation of the integral (see their Eq.(6.20))

$$\int \frac{d^4p}{(2\pi)^4} e^{-ip_\mu(x'^\mu - x^\mu)} \frac{i\varpi}{p_\mu p^\mu - m^2 + i\varepsilon} \mathcal{G}(\mathbf{p}). \qquad (1)$$

The calculation is done in an inertial reference frame[1] $\mathbf{e}_0 = \partial/\partial t$ with arbitrary spatial axes $\langle \mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = \frac{\partial}{\partial y}, \mathbf{e}_3 = \frac{\partial}{\partial z}\rangle$ chosen in such a way that together with $\mathbf{e}_0$ defines a global orthonormal tetrad in Minkoswki spacetime. We next introduce spherical coordinates associated with the selected orthonormal triad $\langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\rangle$ and write

$$\mathbf{p} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta).$$

Then[2]

$$\mathcal{G}(\mathbf{p}) := \gamma^5\gamma^\mu n_\mu, \qquad (2)$$

where the spacelike vector field $n = n^\mu \mathbf{e}_\mu$ is

$$n_\mu := (0, \mathbf{n}), \quad \mathbf{n} := \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\mathbf{p}}{|\mathbf{p}|}\right) = (-\sin\varphi, \cos\varphi, 0)$$

Then the authors claim:

> If there is no preferred direction, and since we are integrating over all momenta, we are free to choose a coordinate system in which $\mathbf{x}' - \mathbf{x}$ lies in the $\hat{z}$ direction. In this special case, the $\mathbf{p}\cdot(\mathbf{x}' - \mathbf{x})$ depends only on $p(=|\mathbf{p}|)$ and θ, but not on φ. Thus, the only φ-dependence in the whole integrand comes from $\mathcal{G}$ which depends on φ in such a manner that an integral over one period vanishes.

Remark 1 *It is very important to remark that if this integral would result non null the fundamental solution for the Klein-Gordon operator would have an additional term, something that obviously cannot be the case.*

[1] In Relativity theory reference frames are represented by timelike vector fields $\mathbf{Z}$ on the manifold modelling spacetime. In Special Relativity if D is the Levi-Civita connection of the Minkowski metric η, an inertial frame is a timelike vector field $\mathbf{I}$ such that $D\mathbf{I} = 0$. Details can be found, e.g., in [4, 5].

[2] See also Eq.(31) in [2].

2 The Integral of $\mathcal{G}(\mathbf{p})$

On the other hand in [2] authors calculated the anticommutator of an elko spinor field with its canonical momentum getting their Eq.(42), i.e.,

$$\{\Lambda(\mathbf{x},t),\Pi(\mathbf{x},t\} = i\delta(\mathbf{x}-\mathbf{x}')\mathbb{I} + i\int \frac{d^3p}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\mathcal{G}(\mathbf{p}) \tag{3}$$

There they claim:

> "Since the integral on the right hand side of Eq.(42) vanishes only along the $\pm\hat{z}_e$ axis, the preferred axis also becomes the axis of locality".

Let us examine if that claim is correct. Call $|\mathbf{x}-\mathbf{x}'| = \Delta$, and put

$$(\mathbf{x}-\mathbf{x}') = \Delta(\sin\theta_\Delta\cos\varphi_\Delta, \sin\theta_\Delta\sin\varphi_\Delta, \cos\theta_\Delta). \tag{4}$$

Calculation of the integral in the second member of Eq.(3) resumes in the calculation of the following integrals [3]

$$\mathbf{J}(\Delta) = \int_0^\infty drr^2\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\varphi e^{ir\Delta f(\theta,\theta_\Delta,\varphi,\varphi_\Delta)}\sin\varphi, \tag{5}$$

and

$$\mathbf{K}(\Delta) = \int_0^\infty drr^2\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\varphi e^{ir\Delta f(\theta,\theta_\Delta,\varphi,\varphi_\Delta)}\cos\varphi, \tag{6}$$

with $f(\theta,\theta_\Delta,\varphi,\varphi_\Delta) = \sin\theta\cos\varphi\sin\theta_\Delta\cos\varphi_\Delta + \sin\theta\sin\varphi\sin\theta_\Delta\sin\varphi_\Delta + \cos\theta\cos\theta_\Delta$.

We will now calculate the integrals in Eqs.(5) and (6) in the cases when $\mathbf{x}-\mathbf{x}'$ lies respectively in the $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ directions. We will call the respective integrals $\mathbf{J}_i(\Delta)$ and $\mathbf{K}_i(\Delta)$, for $i = 1,2,3$.

We start with the observation that it is trivial to verify that $\mathbf{J}_3(\Delta) = 0$ and $\mathbf{K}_3(\Delta) = 0$. To continue we calculate $\mathbf{J}_2(\Delta)$.

So, let us choose the spatial axis such that $(\mathbf{x}-\mathbf{x}') = \Delta\mathbf{e}_2 = \Delta(0,1,0)$ and perform the non-trivial exercise of calculating the value of the integral given by Eq.(5) in this case, i.e.,

$$\mathbf{J}_2(\Delta) = \int_0^\infty drr^2\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\varphi\sin\varphi e^{ir\Delta\sin\theta\sin\varphi}. \tag{7}$$

[3] We take $\Delta \neq 0$. For $\Delta = 0$ it is obvious that $\mathbf{J}(0) = \mathbf{K}(0) = 0$.

We start evaluating the φ-integral,

$$\mathbf{\Omega}(\xi) = \int_0^{2\pi}d\varphi\sin\varphi e^{i\xi\sin\varphi}, \tag{8}$$

where $\xi := r\Delta\sin\theta$. Observe that $\Omega(\xi) = d\Lambda(\xi)/d\xi$ where (see 8.473-4, page 968 of [3])

$$\begin{aligned}\Lambda(\xi) &= \int_0^{2\pi}d\varphi e^{i\xi\sin\varphi} = 2\int_0^\pi d\varphi\cos(r\Delta\sin\theta\sin\varphi)\\ &= 2\pi J_0(r\Delta\sin\theta),\end{aligned} \tag{9}$$

with J_0 the zero order Bessel function. So,

$$\Omega(\xi) = 2\pi i J_1(r\Delta\sin\theta)] \tag{10}$$

where J_1 is the first order Bessel function. Next we evaluate

$$\mathbf{\Xi}(r,\Delta) = 2\pi i\int_0^\pi d\theta\sin\theta J_1(r\Delta\sin\theta). \tag{11}$$

Using the relation 6.681-8, page 739 of [3], namely

$$\int_0^\pi dx\sin(2\mu x)J_{2\nu}(2a\sin x) = \pi\sin(\mu\pi)J_{\nu-\mu}(a)J_{\nu+\mu}(a), \tag{12}$$

valid for $\mathrm{Re}(\nu) > -1$ we see that identifying $2\mu = 1$, $2\nu = 1$ and $2a = r\Delta$ we can write

$$\mathbf{\Xi}(r,\Delta) = 2\pi^2 i J_0\left(\frac{r\Delta}{2}\right)J_1\left(\frac{r\Delta}{2}\right). \tag{13}$$

So, putting $t = r\Delta/2$ we have

$$\mathbf{J}_2(\Delta) = \frac{16\pi^2 i}{\Delta^3}\int_0^\infty dtt^2J_0(t)J_1(t). \tag{14}$$

Now, recall relation 6.626-3, page 715 of [3] (with $\beta = 1$), namely

$$\int_0^\infty dxe^{-2\alpha x}xJ_0(x)J_1(x) = \frac{1}{2\pi}\left[\frac{K\left(\frac{1}{\sqrt{1+\alpha^2}}\right) - E\left(\frac{1}{\sqrt{1+\alpha^2}}\right)}{\sqrt{1+\alpha^2}}\right]. \tag{15}$$

Then we see that

$$\mathbf{J}_2(\Delta) = -\frac{4\pi i}{\Delta^3}\lim_{\alpha\to 0}\frac{d}{d\alpha}\left[\frac{K\left(\frac{1}{\sqrt{1+\alpha^2}}\right) - E\left(\frac{1}{\sqrt{1+\alpha^2}}\right)}{\sqrt{1+\alpha^2}}\right]. \tag{16}$$

Recalling moreover relations 8.113 and 8.114, page 905 of [3] we have for the elliptic functions K and E

$$K\left(\frac{1}{\sqrt{1+\alpha^2}}\right)=\frac{\pi}{2}\ {}_1F_2\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\right),$$
$$E\left(\frac{1}{\sqrt{1+\alpha^2}}\right)=\frac{\pi}{2}\ {}_1F_2\left(-\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\right) \quad (17)$$

where ${}_2F_1$ are Gauss hypergeometric functions. So,

$$\int_0^\infty dx e^{-2\alpha x} x J_0(x) J_1(x)$$
$$=\frac{1}{4}\left[\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\right)-\ {}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\right)}{\sqrt{1+\alpha^2}}\right] \quad (18)$$

and to end our long calculation we must evaluate the limit when $a\rightarrow 0$ of

$$\frac{d}{d\alpha}\left[\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\right)-\ {}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\right)}{\sqrt{1+\alpha^2}}\right]. \quad (19)$$

Recalling moreover that (see, e.g., page 281 of [6])

$$\frac{d}{dz}\ {}_2F_1(a,b;c;z)=\frac{ab}{c}\ {}_2F_1(a+1,b+1;c+1;z),$$

and that ${}_2F_1\left(a,b;c;z\right) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n}\frac{z^n}{n!}$ (with $(a)_n,(b)_n,(c)_n$ the Pochhammer symbols) we see that we are justified in writing

$$\lim_{\alpha\rightarrow 0}\frac{d}{d\alpha}\left\{\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\right)}{\sqrt{1+\alpha^2}}\right\}$$
$$=\lim_{\alpha\rightarrow 0}\left\{\begin{array}{c}-\alpha(1+\alpha^2)^{-3/2}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\right)\\ -2\alpha(1+\alpha^2)^{-5/2}\ {}_2F_1'\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\right)\end{array}\right\}=0. \quad (20)$$

Also,

$$\lim_{\alpha\rightarrow 0}\frac{d}{d\alpha}\left[(1+\alpha^2)^{-1/2}\ {}_2F_1\left(-\frac{1}{2},\frac{1}{2};1;\frac{1}{1+\alpha^2}\right)\right]=0. \quad (21)$$

Finally, using Eqs.(20) and (21) in Eq.(16) we have that $\mathbf{J}_2(\Delta)=0$.

We now evaluate

$$\mathbf{K}_2(\Delta)=\int_0^\infty drr^2\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\varphi e^{ir\Delta\sin\theta\sin\varphi}\cos\varphi. \quad (22)$$

Calling $\boldsymbol{x}=\sin\varphi$ and $r\Delta\sin\theta=\alpha$ we see that (taking into account relation 3.715-9, page 401 of [3] with $n=1$) we have

$$\int_0^{2\pi}d\varphi e^{ir\Delta\sin\theta\sin\varphi}\cos\varphi=\int_{-\pi}^{\pi}d\varphi e^{ir\Delta\sin\theta\sin\varphi}\cos\varphi$$
$$=-2\int_0^\pi d\boldsymbol{x}\cos\varphi\ \cos(\boldsymbol{x}\sin\varphi)=-2[1+(-1)]\frac{\pi}{2}J_1(\boldsymbol{x})=0 \quad (23)$$

and we conclude that $\mathbf{K}_2(\Delta)=0$.

We now evaluate $\mathbf{J}_1(\Delta)$ and $\mathbf{K}_1(\Delta)$. Observe that

$$\int_0^{2\pi}d\varphi\sin\varphi e^{ir\Delta\sin\theta\cos\varphi}=-\int_{-\pi}^{\pi}\sin\varphi e^{ir\Delta\sin\theta\cos\varphi}=0, \quad (24)$$

since the integrand is and odd function and the interval is symmetric. So, we have

$$\mathbf{J}_1(\Delta)=\int_0^\infty drr^2\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\varphi\sin\varphi e^{ir\Delta\sin\theta\cos\varphi}=0. \quad (25)$$

It remains to evaluate

$$\mathbf{K}_1(\Delta)=\int_0^\infty drr^2\int_0^\pi d\theta\sin\theta[\int_0^{2\pi}d\varphi e^{ir\Delta\sin\theta\cos\varphi}\cos\varphi] \quad (26)$$

Writing $\boldsymbol{x}=ir\Delta\sin\theta$ and changing $\varphi\mapsto\varphi+\pi$ the the integral in the brackets in Eq.(26) becomes $2\pi i J_1(r\Delta\sin\theta)$. When this is inserted in Eq.(26) and steps analogous to the ones used in the evaluation of $\mathbf{J}_2(\Delta)$ are used we get $\mathbf{K}_1(\Delta)=0$.

3 Conclusions

We have shown through explicitly and detailed calculation that the integral of $\mathcal{G}(\mathbf{p})$ appearing in Eq.(42) of [2] is null for $\mathbf{x}-\mathbf{x}'$ lying in three orthonormal spatial directions in the rest frame of an arbitrary inertial frame $\mathbf{e}_0=\partial/\partial t$.

This shows that the existence of elko spinor fields does not implies in any breakdown of locality concerning the anticommutator of $\{\Lambda(\mathbf{x},t),\Pi(\mathbf{x}',t\}$ and moreover does not implies in any preferred spacelike direction field in Minkowski spacetime.

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